# AN ANALITICAL METHOD FOR CONSTRUCTION OF SINGLE PARTICLE ELECTRON TRAJECTORIES IN FREE ELECTRON LASERS

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## ABSTRACT

We apply a method of Linshtedt, also called improved expansion, to solve the equations of motion and obtain single-particle trajectories of electrons moving in crossed static magnetic fields of a hybrid non-relativistic free electron laser. Making use of a natural small parameter, the ratio of the amplitude of spatially periodic magnetic field and the guide magnetic field, one can re-write the motion equations for an electron in a form, which allows their solution by an asymptotic series. In such a way the non-linear frequency shifts and renormalized mean electron velocity are calculated analytically. The analytical results are in a good compared with numerical simulations of the electron trajectories.

## INTRODUCTION

Initial analysis of properties of an electron-optical system (EOS) is performed in the approximation of geometrical optics. Namely, for unneutralised electron beam in external magnetic fields it customary to study in succession single-particle approximation, approximation of magneto-hydrodynamics and, finally, kinetic equation formalism. It turns out that even rough single-particle approximation can provide valuable insights into basic dynamics of an electron beam propagating along the studied EOS.

Attempts to find analytical expressions for description of single particle trajectories go back to the 1970s [1–3]. However, no one approach, going further the zero-approximation expression for the helical magnetic field setup [4] and the well-known formulas for the equations of mathematical pendulum in the case of absence of the guide magnetic field, has been presented [5]. This situation is rather unfortunate, since in the case of a free electron laser (FEL) with a giuding magnetic field [6,7] at non-relativistic and intermediate energies (< 600 keV) such an expression could be very helpful. It can provide a qualitative analysis because of relative accessibility of the necessary values of the guiding magnetic fields for the utilization of resonances on the characteristic pump field and cyclotron frequencies.

In the present contribution we present an analytical solution by an asymptotic series of the problem of an electron motion in the harmonic transversal undulator magnetic field and strong, but finite, longitudinal magnetic field.

## IMPROVED EXPANSION

Consider the electron motion in the following static magnetic field:

$$\vec{H} = [0, -H_\perp \sin(2\pi z/l), -H_\parallel].$$

Here $l$ is the space period of the transverse to the injection direction static magnetic field. Non-relativistic equations of motion of an electron take the form

$$m_0 \frac{d\vec{v}}{dt} = -\frac{e}{c}[\vec{v}, \vec{H}],$$

where $m_0$ is the electron's rest mass, $e$ is the absolute value of its charge and $\vec{v}$ is the electron's velocity vector. Let us also introduce the following notations:

$$\tau = \omega_0 t,\ \gamma_0 = \omega_\parallel / \omega_0,\ \varepsilon = \omega_\perp / \omega_0,\ \omega_0 = 2\pi v_\parallel / l,\ \omega_\parallel = eH_\parallel / m_0 c,\ \omega_\perp = eH_\perp / m_0 c,$$

$$\xi = x/l,\ \eta = y/l,\ \zeta = z/l,$$

where $v_\parallel$ is the z-component of the initial velocity of the electrons, $d\zeta/d\tau[\tau=0] = 1/2\pi$ and the rest of the initial values of the velocities and coordinates are equal to zero. Thus we can write the non-linear equations of motion in the dimensionless form

$$\frac{d^2\xi}{d\tau^2} + \gamma_0^2 \xi = -\varepsilon \frac{d\zeta}{d\tau}\sin(2\pi\zeta),\quad \frac{d\zeta}{d\tau} - \delta_0 = \varepsilon \int_0^\tau \frac{d\xi}{d\tau}\sin(2\pi\zeta)d\tau,\quad \frac{d\eta}{d\tau} + \gamma_0 \xi = 0. \quad (1)$$

Here $\gamma_0$ is the dimensionless oscillation frequency and $\delta_0 = d\zeta/d\tau[\tau=0]$. It is easily seen that the first two equations of the system (1) define the dynamics of electrons completely.

We are interested in analytical solutions for the dimensionless frequency $\gamma_0$ greater than 2 and the values of $\varepsilon$ less than ½. According to the Linshtedt method [8,9], we shall expand $\xi(\tau)$, $\eta(\tau)$, $\zeta(\tau)$, the true nonlinear-shifted frequency $\gamma$ and mean electron velocity $\delta$ into series in $\varepsilon$:

$$\xi(\tau) = \xi_0(\gamma\tau) + \varepsilon\xi_1(\gamma\tau) + \varepsilon^2\xi_2(\gamma\tau) + \varepsilon^3\xi_3(\gamma\tau) + \varepsilon^4\xi_4(\gamma\tau) + \ldots$$

$$\zeta(\tau) = \zeta_0(\gamma\tau) + \varepsilon\zeta_1(\gamma\tau) + \varepsilon^2\zeta_2(\gamma\tau) + \varepsilon^3\zeta_3(\gamma\tau) + \varepsilon^4\zeta_4(\gamma\tau) + \ldots,$$

$$\gamma = \gamma_0(1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \varepsilon^4 f_4 + \ldots)$$

$$\delta = \delta_0(1 + \varepsilon g_1 + \varepsilon^2 g_2 + \varepsilon^3 g_3 + \varepsilon^4 g_4 + \ldots)$$

Equating coefficients at each order of $\varepsilon$ for the system of equations (1), we find linearized sets for the functions and coefficients of (2). These sets are iterative linear non-homogeneous systems of equations, which are integrated one after another.

To $o(\varepsilon^3)$ order the solutions have the form

$$\xi(\tau) = \frac{2\pi\delta^2\varepsilon}{\gamma[\gamma^2 - (2\pi\delta)^2]}\{\sin[\gamma\tau] - (\gamma/2\pi\delta)\sin[2\pi\delta\tau]\},$$

$$\eta(\tau) = -\frac{2\pi\delta^2\varepsilon}{\gamma[\gamma^2 - (2\pi\delta)^2]}\{1 - \cos[\gamma\tau] - (\gamma/2\pi\delta)^2(1 - \cos[2\pi\delta\tau])\}, \quad (2)$$

$$\zeta(\tau) = \delta\tau + \frac{\pi\delta^2\varepsilon^2}{\gamma^2 - (2\pi\delta)^2}\left\{\frac{\sin[4\pi\delta\tau]}{16\pi^2\delta^2} - \frac{\sin[(\gamma+2\pi\delta)\tau]}{\gamma+2\pi\delta} + \frac{\sin[(\gamma-2\pi\delta)\tau]}{\gamma-2\pi\delta}\right\}.$$

The frequency $\gamma$ and the average electron velocity $\delta$, up to the same order, are

$$\gamma = \gamma_0 \left[ 1 + \varepsilon^2 \frac{\gamma_0^2 + 1}{4(\gamma_0^2 - 1)^2} \right], \quad \delta = \delta_0 \left[ 1 - \varepsilon^2 \frac{\gamma_0^2 + 3}{4(\gamma_0^2 - 1)^2} \right]. \tag{3}$$

We have accomplished the calculations to the next meaningful order, $o(\varepsilon^5)$. They show that there exist a number of resonances at odd ratios

$$\boxed{\omega_{\parallel} / \omega_0 \approx \gamma / 2\pi\delta = 2k + 1, \; k \in Z.} \tag{4}$$

In the limit $\gamma_0 \to 0$ formulas (2) and (3) provide the trajectories in the FEL without the guide magnetic field (e.g. [5, p. 37]). They as well confirm the assertions on the form of one-body trajectories of electrons in such ideal FEL magnetic field, which are usually made in the literature (cf. [1-3]).

Since in the trajectory approximation the vector potential and components of electromagnetic field in the wave zone are dependent on the time derivatives of the electron coordinates, one can expect that resonances existing in those expressions will also be present for the power of the spontaneous emission. These higher resonances one would expect to observe for experimentally accessible values of the guide magnetic field.

Numerical simulations accomplished by us for the initial system of integro-differential equations (1) verified these analytic solutions to the accuracy of 1%.

## CONCLUSIONS

As a result of this work, we are able to provide analytical solutions for electron trajectories in an ideal hybrid free electron laser-oscillator and calculate dependence of the trajectories on the parameters of the pumping magnetic fields. We have also prepared the machinery for treating the real undulator magnetic field. Thus one will be able to calculate the polar pattern of the emitted radiation in the single particle approximation in the both cases.

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